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On Some Approximation Properties of Real Hardy Spaces ($0 < p \leqslant 1$)

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Atomic decompositions and molecules are used to prove some inequalities of approximation theory in the real Hardy spaces $\operatorname{Re} H_p$ defined on the one-dimensional torus T or on $\mathbb{R}$, $0 < p \leqslant 1$. Considerations are mainly based on a description of the $\operatorname{Re} H_p$-moduli of continuity by a corresponding K'-functional. In particular, inequalities of Jackson type are obtained for spline approximation in the periodic case and for Bochner–Riesz summability in the case of $\mathbb{R}$.

0. INTRODUCTION

Let $H_p(D)$ $(0 < p < \infty)$ be the complex quasi Banach space of analytic functions $F(z)$ on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ for which $\|F\|_{H_p(D)} = \sup_{r<1} \{(1/2\pi) \int_{-\pi}^{\pi} |F(re^{it})|^p \, dt\}^{1/p} < \infty$. These spaces were introduced by G. H. Hardy and F. Riesz and played an important role in the investigation of the boundary behaviour of analytic and harmonic functions and in Fourier analysis (cf. [8, 15, 31], where the basic properties of $H_p(D)$ are described). For instance, if $F(z) \in H_p(D)$ for some $p > 0$, then there exists a.e. on the torus $T = (-\pi, \pi]$ the limit $F(e^{it}) = \lim_{r \to 1-} F(re^{it})$, and $|F(e^{it})|$ belongs to the Lebesgue space $L_p(T)$ of all real-valued, 2π-periodic measurable functions $f(t)$ satisfying $\|f\|_p = \{(1/2\pi) \int_{-\pi}^{\pi} |f(t)|^p \, dt\}^{1/p} < \infty$. As usual, $L_\infty(T)$ denotes the space of all bounded measurable 2π-periodic real functions with the corresponding norm.

Now the definition of $\operatorname{Re} H_p(T)$ will be given. A real-valued distribution $u(t) \in \mathscr{D}'(T)$ belongs to $\operatorname{Re} H_p(T)$ $(0 < p < \infty)$ iff there exists a function $F(z) \in H_p(D)$ with the properties $\operatorname{Im} F(0) = 0$ and $u(t) = \lim_{r \to 1-} \operatorname{Re} F(re^{it})$ in the sense of distributions (if $p \geqslant 1$ then $\operatorname{Re} H_p(T)$ can be treated as a subspace of $L_p(T)$, and $u(t) = \operatorname{Re} F(e^{it})$). Equipped with the quasi norm $\|u\|_{\operatorname{Re} H_p(T)} = \|F\|_{H_p(D)}$ the class $\operatorname{Re} H_p(T)$ obviously becomes a real quasi Banach space with quite the same properties as $H_p(D)$. It is well known that for $1 < p < \infty$ $\operatorname{Re} H_p(T)$ coincides with $L_p(T)$, while for $0 < p \leqslant 1$ $\operatorname{Re} H_p(T)$

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and $L_p(T)$ lead to different scales of function spaces. In the following it is this case that will be considered.

The by now classical investigations by G. H. Hardy, F. and M. Riesz, J. E. Littlewood and others of $H_p(D)$ (and $\operatorname{Re} H_p(T)$) employed complex methods and showed that, for several problems in Fourier analysis, the use of the H_p-scale ($0 < p \leqslant 1$) is preferable over that of the L_p-scale. During the past 15 years a powerful theory of H_p spaces ($0 < p \leqslant 1$), especially in the n-dimensional case, has been developed by means of real methods (maximal functions, decomposition techniques, atoms, molecules, etc.), and various new applications to Fourier analysis and singular operators have been given; cf. [6, 9, 27, 28].

However, concerning approximation properties (for instance, inequalities of Jackson type) only few results are known. We refer to the papers by E. A. Storoženko [21–25], where special integral representations and complex techniques have been used to obtain estimates of the following type. Let $F(z) \in H_p(D)$, $0 < p \leqslant 1$, and let $\sigma_n^\alpha F(z)$ be the nth (C, α)-mean of the power series of $F(z)$. Then, for $n = 1, 2, \ldots$, we have [23]

$$\|F - \sigma_n^\alpha F\|_{H_p(D)} \leqslant C_{p,\alpha} \cdot \omega\left(\frac{\pi}{n}, F\right)_{H_p} \begin{cases} 1, & \alpha > 1/p - 1, \\ (\ln n)^{1/p}, & \alpha = 1/p - 1, \\ n^{1/p - 1 - \alpha}, & -1 < \alpha < 1/p - 1, \end{cases} \tag{0.1}$$

where $\omega(\delta, F)_{H_p} = \sup_{0 \leqslant h \leqslant \delta} \|F(z) - F(ze^{ih})\|_{H_p(D)}$, $0 \leqslant \delta \leqslant \pi$, denotes the corresponding modulus of continuity (here and in the following C, C_p, $C_{p,\alpha}, \ldots$, denote positive constants depending on the cited parameters only and changing their concrete values from line to line). Analogous results were established for other summation methods (cf. [21–24]). In [23, 25] the Jackson-type inequality

$$\inf_{P_n(z) = \sum_{j=0}^n a_j z^j} \|F - P_n\|_{H_p(D)} \leqslant C_{p,k,l}(n+1)^{-l} \cdot \omega_k\left(\frac{\pi}{n+1}, F^{(l)}\right)_{H_p} \tag{0.2}$$

($n = 0, 1, \ldots, F^{(l)}(z) \in H_p(D)$, $0 < p \leqslant 1$, $l = 0, 1, \ldots$) with moduli of continuity of arbitrary order $k = 1, 2, \ldots$, was proved. Some generalizations of (0.1), (0.2) to Hardy spaces on the polydisc were given by J. Valašek [29].

In this paper we intend to prove some further approximation properties in the spaces $\operatorname{Re} H_p(T)$, $0 < p \leqslant 1$, by using atomic representations and molecules. Our considerations are based on the relation

$$\omega_k(\delta, u)_{\operatorname{Re} H_p} \overset{k,p}{\asymp} \inf_{g^{(k)}(t) \in \operatorname{Re} H_p(T)} \{\|u - g\|_{\operatorname{Re} H_p(T)} + \delta^k \| g^{(k)} \|_{\operatorname{Re} H_p(T)}\}, \tag{0.3}$$

where $u(t) \in \operatorname{Re} H_p(T)$, $0 < p \leqslant 1$, $k = 1, 2, \ldots$, and $\delta \in [0, \pi]$. This two-sided inequality is the analog of a well-known assertion for $L_p(T)$, $1 \leqslant p < \infty$, cf. [10, 14], the notation $A \asymp^{\alpha, \beta, \ldots} B$ means both $A \leqslant C_{\alpha,\beta,\ldots} \cdot B$ and $B \leqslant C_{\alpha,\beta,\ldots} \cdot A$. The proof of (0.3) will be furnished in Section 2, in particular, we use inequality (0.2) for $l = 0$.

In Section 3 a Jackson-type inequality will be established for the approximation of $u(t) \in \operatorname{Re} H_p(T)$ by the partial sums $P_n^{(m)} u(t)$ of its Fourier series with respect to the periodic orthonormal spline systems $F^{(m)}$ $(m = 0, 1, \ldots,)$ introduced by Z. Ciesielski (cf. [2–4], our notations are different from those given in these papers, for details see Section 1). Recently, we have proved the systems $F^{(m)}$ to be Schauder bases in $\operatorname{Re} H_p(T)$ for $(m+1)^{-1} \leqslant p \leqslant 1$ (see [12, 13]). Analogous results have independently been obtained by P. Sjölin, J.-O. Strömberg [17, 18], and P. Wojtaszczyk [30] (in addition, these authors proved unconditional convergence for $p > (m+1)^{-1}$).

The inequalities

$$\| u - P_n^{(m)} u \|_{\operatorname{Re} H_p(T)} \leqslant C_m \cdot \omega_{m+1}(\pi/n, u)_{\operatorname{Re} H_p}, \qquad n = 1, 2, \ldots, \tag{0.4}$$

given in this paper (cf. Theorem 2 in Section 3) estimate the rate of convergence of the basis expansion for $u(t) \in \operatorname{Re} H_p(T)$, $(m+1)^{-1} \leqslant p \leqslant 1$, $m = 0, 1, \ldots$, and extend the corresponding assertions for $L_p(T)$, $1 \leqslant p < \infty$, and $C(T)$ due to Z. Ciesielski [3].

Furthermore, in Section 3 some properties of best spline approximation in the classes $\operatorname{Re} H_p(T)$, $0 < p \leqslant 1$, are discussed (concerning best spline approximation in L_p spaces for $p < 1$, cf. [11]).

Finally, it should be mentioned that our approach can be used for the Hardy spaces $\operatorname{Re} H_p(\mathbb{R})$, $0 < p \leqslant 1$, on the real line, too. Without going into detail, an application to the approximation by Bochner–Riesz means of the Fourier integral of distributions belonging to $\operatorname{Re} H_p(\mathbb{R})$ will be given in Theorem 3 at the end of Section 3.

1. Preliminaries

Hardy spaces

First the atomic characterization of $\operatorname{Re} H_p(T)$, $0 < p \leqslant 1$, will be described. A function $a(t) \in L_q(T)$, $1 \leqslant q \leqslant \infty$, is called (p, q, s)-atom centered at $t_0 \in T$ if $p < q$, the integer s satisfies $s \geqslant [1/p - 1]$, and

$$\begin{gathered} \operatorname{supp} a(t) \subset J = (t_0 - |J|/2,\, t_0 + |J|/2], \qquad |J| \leqslant 2\pi, \\ \|a\|_q \leqslant |J|^{1/q - 1/p}, \\ \int_{t_0 - \pi}^{t_0 + \pi} a(t) \cdot (t - t_0)^l \, dt = 0, \qquad l = 0, \ldots, s. \end{gathered} \tag{1.1}$$

PROPOSITION 1. *Let* $0 < p \leqslant 1$, $1 \leqslant q \leqslant \infty$, $p < q$, *and* $s \geqslant [1/p - 1]$.

(a) *If* $u(t) \in \operatorname{Re} H_p(T)$ *then there exists a decomposition*

$$u(t) = \sum_{j=0}^{\infty} \lambda_j \cdot a_j(t) \text{ (convergence in } \mathbb{D}'(T)), \tag{1.2}$$

where the λ_j *are reals satisfying* $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$, *the* $a_j(t)$ *are* (p, q, s)*-atoms for* $j = 1, 2, \ldots$, $a_0(t) \in L_q(T)$ *with* $\|a_0\|_q \leqslant 1$, *and*

$$\left\{ \sum_{j=0}^{\infty} |\lambda_j|^p \right\}^{1/p} \leqslant C_{p,q,s} \cdot \|u\|_{\operatorname{Re} H_p(T)}. \tag{1.3}$$

(b) *Conversely, if* λ_j *and* $a_j(t)$ *are as assumed above then the right-hand side of* (1.2) *converges in the quasi norm of* $\operatorname{Re} H_p(T)$ *to a certain* $u(t) \in \operatorname{Re} H_p(T)$, *and*

$$\|u\|_{\operatorname{Re} H_p(T)} \leqslant C_{p,q,s} \left\{ \sum_{j=0}^{\infty} |\lambda_j|^p \right\}^{1/p}. \tag{1.4}$$

The proof of this result is essentially due to R. R. Coifman [5] (cf. also [6, 27], the considerations in the periodic case are quite the same as for $\mathbb{R}$ or $\mathbb{R}^N$, $N > 1$).

Following [6, 27], a function $m(t) \in L_q(T)$ is said to be a (p, q, s)-molecule centered at $t_0 \in T$ if there exists some $\varepsilon > \max(s, 1/p - 1)$ so that with $\alpha = 1 - 1/p + \varepsilon$ and $\beta = 1 - 1/q + \varepsilon$ we have

$$\begin{gathered} N(m) = \|m\|_q^{\alpha/\beta} \cdot \|m(t) \cdot d_T(t, t_0)^\beta\|_q^{1-\alpha/\beta} < \infty, \\ \int_{t_0-\pi}^{t_0+\pi} m(t) \cdot (t - t_0)^l \, dt = 0, \quad l = 0, \ldots, s. \end{gathered} \tag{1.5}$$

(By $d_T(t, t')$ we denote the periodic distance of $t, t' \in T$, i.e., if t, t' are taken in the interval $(-\pi, \pi]$ then $d_T(t, t') = \min(|t - t'|, 2\pi - |t - t'|)$.)

PROPOSITION 2. *Let* p, q, s *be as above. If* $m(t)$ *is a* (p, q, s)*-molecule then* $m(t) \in \operatorname{Re} H_p(T)$, *and*

$$\|m\|_{\operatorname{Re} H_p(T)} \leqslant C_{p,q,\varepsilon} \cdot N(m). \tag{1.6}$$

The proof of (1.6) runs as in [27] where the case of $\mathbb{R}^N$ is considered. It should be mentioned that for applications the most interesting cases are $q = \infty$, $q = 2$, and $q = 1$.

Spline Systems

Let $n = 1, 2, \ldots$. The dyadic partitions

$$\pi_n = \{-\pi < s_{n,1} < \cdots < s_{n,n-1} < s_{n,n} = \pi\}$$

of T are defined by setting

$$
\begin{aligned}
s_{n,i} &= -\pi + \pi \cdot 2^{-k} \cdot i, && i = 1,\dots, 2l, \\
&= \pi - \pi \cdot 2^{1-k} \cdot (n-i), && i = 2l+1,\dots, n,
\end{aligned}
$$

for $n = 2^k + l > 1$, where $l = 1,\dots, 2^k$ and $k = 0, 1,\dots$. Furthermore, we put $s_{n,jn+i} = s_{n,i}$, $s'_{n,jn+i} = s_{n,i} + 2\pi \cdot j$ for $i = 1,\dots, n$ and $j = 0, \pm 1,\dots$.

For $m = 0, 1,\dots$, we denote by $S_n^{(m)}(T)$ the n-dimensional subspace of $C^{(m-1)}(T)$ (of $L_\infty(T)$ if $m = 0$) consisting of all periodic spline functions of degree m with respect to π_n, i.e., $g(t) \in S_n^{(m)}(T)$ whenever $g(t)$ coincides with a certain algebraic polynomial of degree $\leqslant m$ on each interval $(s_{n,i-1}, s_{n,i}]$ and belongs to $C^{(m-1)}(T)$ if $m > 0$. The corresponding B-splines

$$
N_{n,i}^{(m)}(t) = \sum_{j=-\infty}^{\infty} N'^{(m)}_{n,jn+i}(t), \qquad t \in (-\pi, \pi], i = 1,\dots, n,
$$

where $N'^{(m)}_{n,j}(t) = (s'_{n,j+m+1} - s'_{n,j}) \cdot [s'_{n,j},\dots, s'_{n,j+m+1}; (s-t)_+^m]$, $-\infty < t < \infty$, have the properties (cf. [7])

$$
\begin{aligned}
\operatorname{supp} N_{n,i}^{(m)}(t) &= T, \qquad n = 1,\dots, m+1, \\
&= (s_{n,i}, s_{n,i+m+1}] = \Delta_{n,i}^{(m)}, \qquad n > m+1,
\end{aligned}
\tag{1.7}
$$

$$
\sum_{i=1}^{n} N_{n,i}^{(m)}(t) = 1, \quad N_{n,i}^{(m)}(t) \geqslant 0, \quad t \in T,\ i = 1,\dots, n \tag{1.8}
$$

(the intervals have to be understood as intervals defined on the torus T). Furthermore, the system $\{N_{n,i}^{(m)}(t)\}$, $i = 1,\dots, n$, forms an algebraic basis in $S_n^{(m)}(T)$. By $\{\underline{N}_{n,i}^{(m)}(t)\}$, $i = 1,\dots, n$, we denote its biorthogonal system in $S_n^{(m)}(T)$ with respect to the scalar product $(\cdot\,,\cdot)$ in $L_2(T)$. From a result of J. Domsta [7] it easily follows that

$$
|\underline{N}_{n,i}^{(m)}(t)| \leqslant C_m \cdot n \cdot q^{n \cdot d_T(s_{n,i},t)}, \qquad t \in T, i = 1,\dots, n \tag{1.9}
$$

holds with some constant q $(0 < q < 1)$ only depending on m.

For fixed $m = 0, 1,\dots$, the periodic orthonormal spline system $F^{(m)} = \{f_n^{(m)}(t)\}$, $n = 1, 2,\dots$, is uniquely determined by the conditions

$$
\begin{gathered}
f_n^{(m)}(t) \in S_n^{(m)}(T), \qquad n = 1, 2,\dots \\
F^{(m)} \text{ is an orthonormal system in } L_2(T), \\
f_n^{(m)}(s_{n,2l-1}) > 0, \quad n = 2^k + l \geqslant 2, f_1^{(m)}(t) = (2\pi)^{-1/2}.
\end{gathered}
\tag{1.10}
$$

Concerning this definition and the basic properties of $F^{(m)}$ and its nonperiodic counterpart we refer to the papers of Z. Ciesielski a.o. (cf.

[2–4, 7], where the notations are somewhat different from those given here). For instance, $F^{(m)}$ forms a Schauder basis in $L_p(T)$, $1 \leqslant p < \infty$. The partial sums $P_n^{(m)}f(t)$ of the Fourier series with respect to $F^{(m)}$ of a function $f(t) \in L_2(T)$ can be written in the form

$$P_n^{(m)}f(t) = \sum_{j=1}^{n} (f_j^{(m)}, f) \cdot f_j^{(m)}(t)$$

$$= \sum_{j=1}^{n} (N_{n,j}^{(m)}, f) \cdot N_{n,j}^{(m)}(t), \qquad n \geqslant 1. \tag{1.11}$$

Some further properties of $F^{(m)}$ in connection with $\operatorname{Re} H_p(T)$ will be stated in Section 3.

2. Moduli of Continuity

In this Section, let $0 < p \leqslant 1$, $k = 0, 1, \ldots$, and $u(t) \in \operatorname{Re} H_p(T)$ be arbitrary but fixed. The function

$$\omega_k(\delta, u)_{\operatorname{Re} H_p} = \sup_{0 \leqslant h \leqslant \delta} \|\Delta_h^k u(t)\|_{\operatorname{Re} H_p(T)}, \qquad \delta \in [0, \pi], \tag{2.1}$$

where $\Delta_h^k u(t) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \cdot u(t + lh)$, $h \geqslant 0$, is called H_p-modulus of continuity of order k for $u(t)$. Obviously (cf. Section 0), we have

$$\omega_k(\delta, u)_{\operatorname{Re} H_p} = \omega_k(\delta, F)_{H_p}, \qquad \delta \in [0, \pi], \tag{2.2}$$

where $F(z)$ denotes the corresponding analytic function. The basic properties of the H_p-moduli (2.1) are similar to those of the usual L_p-moduli of continuity $\omega_k(\delta, f)_p$ for $f(t) \in L_p(T)$ (for $p \geqslant 1$ cf. [10], for $p < 1$, [11, 26]).

Theorem 1. *Let* $0 < p \leqslant 1$, $k = 1, 2, \ldots$, *and* $u(t) \in \operatorname{Re} H_p(T)$. *Then for* $\delta \in [0, \pi]$ *we have*

$$\omega_k(\delta, u)_{\operatorname{Re} H_p} \overset{k,p}{\asymp} \inf_{g(t) \in \operatorname{Re} H_p^k(T)} \{\|u - g\|_{\operatorname{Re} H_p(T)} + \delta^k \cdot \|D^k g\|_{\operatorname{Re} H_p(T)}\}, \tag{2.3}$$

where

$$\operatorname{Re} H_p^k(T) = \left\{ g(t) \sim \sum_{l=-\infty}^{\infty} c_l e^{ilt} \in \mathscr{D}'(T) : D^k g(t) \right.$$

$$\left. \sim \sum_{l=-\infty}^{\infty} (il)^k c_l e^{ilt} \in \operatorname{Re} H_p(T) \right\}.$$

Proof. First we consider any $g(t)\in \operatorname{Re} H_p^k(T)$. Let $q=\infty$, $s=[1/p-1]+k$, and

$$D^k g(t)=\sum_{j=0}^{\infty}\lambda_j\cdot a_j(t),\qquad \left\{\sum_{j=0}^{\infty}|\lambda_j|^p\right\}^{1/p}\overset{k,p}{\asymp}\|D^k g\|_{\operatorname{Re} H_p(T)},\tag{2.4}$$

be the decomposition of $D^k g(t)$ into (p, q, s)-atoms according to Proposition 1. Since $(a_j, 1)=0$, $j=0, 1,\ldots$, we can introduce unique functions $A_j(t)\in L_\infty(T)$ satisfying the relations $(A_j, 1)=0$ and $a_j(t)=D^k A_j(t)$ in the sense of $\mathscr{D}'(T)$ and a.e. on T. For instance, if $j=1, 2,\ldots$ then

$$A_j(t)=\int_{t_{0,j}-\pi}^{t}\cdots\int_{t_{0,j}-\pi}^{t_{k-1}} a_j(t_k)\,dt_k\cdots dt_1,\qquad t\in(t_{0,j}-\pi, t_{0,j+\pi}],\tag{2.5}$$

where $t_{0,j}$ is the center of the supporting interval J_j of $a_j(t)$ (cf. (1.1)). Moreover, from the assumed facts we easily obtain that the functions $|J_j|^{-k}A_j(t), j=1, 2,\ldots$, are $(p, \infty, [1/p-1])$-atoms with the same supporting intervals J_j, and that $|A_0(t)|\leqslant(2\pi)^k$, $t\in T$. Therefore,

$$g(t)=c_0+\sum_{j=0}^{\infty}\lambda_j\cdot A_j(t)\quad (\text{in the sense of } \mathscr{D}'(T))\tag{2.6}$$

represents a corresponding atomic decomposition for $g(t)$, and

$$\|g\|_{\operatorname{Re} H_p(T)}\leqslant C_p\left\{|c_0|^p+\sum_{j=0}^{\infty}|J_j|^{kp}|\lambda_j|^p\right\}^{1/p}$$
$$<\infty\qquad (J_0=T).\tag{2.6}$$

(Actually, we could show somewhat more, namely, that $g(t)$ belongs to $\operatorname{Re} H_{p/(1-kp)}(T)$ for $kp<1$, and to $C(T)$ for $kp\geqslant 1$. which gives the real variant of a classical assertion of G. H. Hardy and J. E. Littlewood (cf. [8]).)

Furthermore, we have

$$\|\Delta_h^k u\|^p_{\operatorname{Re} H_p(T)}\leqslant\|\Delta_h^k(u-g)\|^p_{\operatorname{Re} H_p(T)}+\|\Delta_h^k g\|^p_{\operatorname{Re} H_p(T)}$$
$$\leqslant C_{k,p}\left\{\|u-g\|^p_{\operatorname{Re} H_p(T)}+\sum_{j=0}^{\infty}|\lambda_j|^p\cdot\|\Delta_h^k A_j\|^p_{\operatorname{Re} H_p(T)}\right\}.\tag{2.7}$$

According to the above considerations on the $A_j(t)$'s we obtain for $|J_j|\leqslant\delta$

$$\|\Delta_h^k A_j\|^p_{\operatorname{Re} H_p(T)}\leqslant C_{k,p}\|A_j\|^p_{\operatorname{Re} H_p(T)}\leqslant C_{k,p}|J_j|^{kp}$$
$$\leqslant C_{k,p}\cdot\delta^{kp}.\tag{2.8}$$

Now, let $\delta < |J_j| \leqslant \pi/(k+1)$. Then we have $(0 \leqslant h \leqslant \delta)$

$$\operatorname{supp} \Delta_h^k A_j(t) \subset J_j' = (t_{0,j} - |J_j|/2 - k\delta,\ t_{0,j} + |J_j|/2 + k\delta],$$
$$\|\Delta_h^k A_j\|_\infty \leqslant h^k \|a_j\|_\infty \leqslant \delta^k \cdot |J_j|^{-1/p} \leqslant C_k \cdot \delta^k \cdot |J_j'|^{-1/p}, \qquad (2.9)$$

and

$$\int_{t_{0,j}-\pi}^{t_{0,j}+\pi} \Delta_h^k A_j(t) \cdot (t - t_{0,j})^l \, dt = 0, \qquad l = 0, \ldots, [1/p - 1].$$

Thus, by Proposition 1 this again yields (2.8) for $h \in [0, \delta]$ at most. Finally, if $|J_j| \geqslant \max(\delta, \pi/(k+1))$ then the obvious inequality

$$\|\Delta_h^k A_j\|_{\operatorname{Re} H_p(T)} \leqslant C_p \cdot \|\Delta_h^k A_j\|_\infty,$$

together with (2.9) give the desired estimate (2.8) for $h \in [0, \delta]$ and arbitrary $j = 0, 1, \ldots$.

Now, from (2.7), (2.8), and Proposition 1 it easily follows that

$$\omega_k(\delta, u)_{\operatorname{Re} H_p} \leqslant C_{k,p} \inf_{g(t) \in \operatorname{Re} H_p^k(T)} \{\|u - g\|_{\operatorname{Re} H_p(T)} + \delta^k \cdot \|D^k g\|_{\operatorname{Re} H_p(T)}\}.$$

Hence (2.3) is established in one direction.

The inverse inequality will be obtained as a corollary to the Jackson-type inequality (0.2) for $H_p(D)$. Let $n = 0, 1, \ldots,$ be defined by the relation $\pi/(n+1) \leqslant \delta < \pi/n$. From (0.2) with $l = 0$ and the definition of $\operatorname{Re} H_p(T)$ it follows that there exist a complex polynomial $P_n(z)(\operatorname{Im} P_n(0) = 0)$ and a real trigonometric polynomial $T_n(t)(= \operatorname{Re} P_n(e^{it}))$, satisfying

$$\|F - P_n\|_{H_p(D)} = \|u - T_n\|_{\operatorname{Re} H_p(T)} \leqslant C_{k,p} \omega_k(\delta, u)_{\operatorname{Re} H_p}. \qquad (2.10)$$

For estimating $\|D^k T_n\|_{\operatorname{Re} H_p(T)}$ we shall use the relation

$$\|D^k T_n\|_p \overset{k,p}{\asymp} (n+1)^{-k} \, \omega_k(\pi/(n+1), T_n)_p, \qquad (2.11)$$

independently obtained for $0 < p < 1$ by E. A. Storoženko [24] and V. I. Ivanov (unpublished). For instance, (2.11) can be proved by appropriately using the Taylor expansion of $T_n(t)$ and the inequality of Bernstein type for $L_p(T)$(cf. [26, Theorem 3.2 and Lemma 3.1]). The details will be omitted. If $\tilde{T}_n(t)(= \operatorname{Im} P_n(e^{it}))$ denotes the conjugate trigonometric polynomial then (cf. (2.11), the relation between $\operatorname{Re} H_p(T)$ and $H_p(D)$, and (2.2))

$$\begin{aligned}\|D^k T_n\|_{\operatorname{Re} H_p(T)}^p &\leqslant \|D^k T_n\|_p^p + \|D^k \tilde{T}_n\|_p^p \\ &\leqslant C_{k,p} \cdot (n+1)^{kp} \{\omega_k(\pi/(n+1), T_n)_p^p + \omega_k(\pi/(n+1), \tilde{T}_n)_p^p\} \\ &\leqslant C_{k,p} \cdot \delta^{-kp} \omega_k(\delta, P_n)_{H_p}^p = C_{k,p} \cdot \delta^{-kp} \omega_k(\delta, T_n)_{\operatorname{Re} H_p}^p.\end{aligned}$$

Thus, we get

$$\delta^k \cdot \|D^k T\|_{\mathrm{Re}\, H_p(T)} \leqslant C_{p,k} \omega_k(\delta, T_n)_{\mathrm{Re}\, H_p} \leqslant C_{k,p} \{\|u - T_n\|_{\mathrm{Re}\, H_p(T)} + \omega_k(\delta, u)_{\mathrm{Re}\, H_p}\},$$

which together with (2.10) gives the desired result. Theorem 1 is completely proved.

Remark 1. Theorem 1 gives an explicit characterization of the K- and K'-functionals for real interpolation between the spaces $\mathrm{Re}\, H_p(T)$ and $\mathrm{Re}\, H_p^k(T)$, $0 < p \leqslant 1$ (for definitions cf. [1, 10]). Results of this kind are well-known for the L_p-spaces, $1 \leqslant p < \infty$ [10, 14]. As an easy consequence, (2.3) yields that for $0 < p \leqslant 1$,

$$(\mathrm{Re}\, H_p(T), \mathrm{Re}\, H_p^k(T))_{\theta,q} = B_{p,q}^{\theta k}(T) = \left\{ g(t) \in \mathrm{Re}\, H_p(T) : \|g\|_{B_{p,q}^{\theta k}} = \|g\|_{\mathrm{Re}\, H_p(T)} + \left\{ \int_0^\pi \frac{\omega_k(t, g)_{\mathrm{Re}\, H_p}^q \, dt}{t^{\theta k q + 1}} \right\}^{1/q} < \infty \right\},$$

where $0 < q < \infty$, $0 < \theta < 1$ (modification if $q = \infty$). This fact seems to be known (cf. [28] for the corresponding R^N-results).

Probably, further applications of Theorem 1 might be given. For example, the following nonobvious property of the H_p-modulus of continuity can immediately be proved by (2.3):

$$\text{If } u(t) \in \mathrm{Re}\, H_p(T), \quad u(t) \neq \text{const.}, \quad 0 < p < 1, \quad \text{then} \quad \omega_k(\delta, u)_{\mathrm{Re}\, H_p} \geqslant C_{p,k,u(t)} \cdot \delta^k, \ \delta \in [0, \pi], \ k = 1, 2, \ldots. \tag{2.12}$$

Thus, the saturation properties of the H_p-moduli of continuity differ from those of the moduli of continuity for the corresponding L_p-spaces, $0 < p < 1$ (cf. [11, 26]).

Remark 2. It would be of some interest to establish (2.3) by real methods only, i.e., without referring to (0.2). In particular, this might open the possibility for obtaining N-dimensional analogs of (2.3) for $H_p(\mathbb{R}^N)$ and $H_p(T^N)$, $0 < p < 1$, with the corresponding applications in approximation theory and related topics.

3. Approximation by Splines

First let us mention

Proposition 3. *Let* $0<p\leqslant 1$, $m=0, 1,\ldots$. *Then the periodic orthonormal spline system* $F^{(m)}$ (cf. (1.10)) *forms a Schauder basis for* $\operatorname{Re} H_p(T)$ *if* $(m+1)^{-1}\leqslant p\leqslant 1$. *More precisely, the partial sum operators* (1.11) *can appropriately be extended to* $\operatorname{Re} H_p(T)$ *and satisfy*

$$\|P_n^{(m)}u\|_{\operatorname{Re} H_p(T)}\leqslant C_m\cdot\|u\|_{\operatorname{Re} H_p(T)},\qquad (m+1)^{-1}\leqslant p\leqslant 1,\quad n=1, 2,\ldots \tag{3.1}$$

Proposition 3 was proved in [12, 13] by using atomic decompositions only. In [17] analogous results (inclusively concerning unconditionality) were obtained for $H_p(0, 1)$, $p>(m+1)^{-1}$, by a combination of atomic and maximal techniques while [30] deals with the periodic case and uses the language of molecules (on the preprint [30] we have been informed only after the preparation of the main parts of this paper, some of its arguments would involve technical simplifications in our proof of Theorem 2).

Naturally, there arises the question of estimating the rate of convergence of the basis expansion with respect to $F^{(m)}$ in the $\operatorname{Re} H_p$ quasi norm. The answer will be given by the following main result of this section.

Theorem 2. *Let* $m=0, 1,\ldots$, $u(t)\in\operatorname{Re} H_p(T)$, *and* $(m+1)^{-1}\leqslant p\leqslant 1$. *Then we have*

$$\|u-P_n^{(m)}u\|_{\operatorname{Re} H_p(T)}\leqslant C_m\cdot\omega_{m+1}(\pi/n, u)_{\operatorname{Re} H_p},\qquad n=1, 2,\ldots \tag{3.2}$$

Proof. According to Theorem 1 the estimate (3.2) follows from the inequalities (3.1) and

$$\|g-P_n^{(m)}g\|_{\operatorname{Re} H_p(T)}\leqslant C_m\cdot n^{-m-1}\cdot\|D^{m+1}g\|_{\operatorname{Re} H_p(T)},\qquad n=1, 2,\ldots, \tag{3.3}$$

where $g(t)\in\operatorname{Re} H_p^{m+1}(T)$, $(m+1)^{-1}\leqslant p\leqslant 1$. Indeed, by these inequalities we have

$$\begin{aligned}\|u-P_n^{(m)}u\|^p_{\operatorname{Re} H_p(T)}&\leqslant\|u-g\|^p_{\operatorname{Re} H_p(T)}+\|g-P_n^{(m)}g\|^p_{\operatorname{Re} H_p(T)}\\&\quad+\|P_n^{(m)}(u-g)\|^p_{\operatorname{Re} H_p(T)}\\&\leqslant C_m\cdot\{\|u-g\|_{\operatorname{Re} H_p(T)}+n^{-m-1}\cdot\|D^{m+1}g\|_{\operatorname{Re} H_p(T)}\}^p\end{aligned}$$

for arbitrary $g(t)\in\operatorname{Re} H_p^{m+1}(T)$. Now, by taking the infimum and from (2.3) we get the desired relation (3.2).

In order to prove (3.3) we need its analog

$$\| g - P_n^{(m)} g\|_\infty \leqslant C_{m \cdot n}{}^{-m-1} \|D^{m+1} g\|_\infty, \qquad n = 1, 2,..., \tag{3.4}$$

for functions $g(t) \in L_\infty(T)$ with absolutely continuous mth derivative and $D^{m+1}g(t) \in L_\infty(T)$. Inequality (3.4) was proved in [3] for the nonperiodic case, this proof also holds with minor changes in the periodic case considered here.

Now fix arbitrary $m = 0, 1,..., p \in [(m+1)^{-1}, 1]$, and $g(t) \in \operatorname{Re} H_p^{m+1}(T)$. Let

$$D^{m+1}g(t) = \sum_{j=0}^{\infty} \lambda_j \cdot a_j(t), \qquad g(t) = c_0 + \sum_{j=0}^{\infty} \lambda_j \cdot A_j(t),$$

be the corresponding atomic decompositions described in Section 2 (cf. (2.4)–(2.6) for $k = m+1$). Obviously, (3.3) will be proved if we verify the inequality

$$\|A_j - P_n^{(m)} A_j\|_{\operatorname{Re} H_p(T)} \leqslant C_m \cdot n^{-m-1}, \quad n = 1, 2,..., j = 0, 1,.... \tag{3.3'}$$

Since (cf. (3.4))

$$\begin{aligned} \|A_j - P_n^{(m)} A_j\|_{\operatorname{Re} H_p(T)} &\leqslant C \cdot \|A_j - P_n^{(m)} A_j\|_\infty \\ &\leqslant C_m \cdot n^{-m-1} \|a_j\|_\infty \leqslant C_m \cdot n^{-m-1} |J_j|^{-1/p}, \end{aligned} \tag{3.5}$$

and by (3.1) we have

$$\|A_j - P_n^{(m)} A_j\|_{\operatorname{Re} H_p(T)} \leqslant C_m \cdot \|A_j\|_{\operatorname{Re} H_p(T)} \leqslant C_m \cdot |J_j|^{m+1} \ (\leqslant C_m),$$

it only remains to check (3.3′) for $n \geqslant n_0$, and $2\pi \cdot n^{-1} \leqslant |J_j| \leqslant C_0$, where the choice of n_0, C_0 will be clear from the considerations below.

Let $a_j(t) = a(t)$ be a (p, ∞, s)-atom ($s = [1/p - 1] + m + 1$) with these properties, i.e., $2\pi \cdot n^{-1} \leqslant |J_j| = |J| \leqslant C_0$, where $J_j = J = (\alpha, \beta] = (t_0 - |J|/2, t_0 + |J|/2]$ denotes the common supporting interval of both $a(t)$ and

$$A_j(t) = A(t) = \int_\alpha^t \int_\alpha^{\xi_m} \cdots \int_\alpha^{\xi_1} a(\xi_0)\, d\xi_0 \cdots d\xi_{m-1}\, d\xi_m.$$

Let $J' = (t_0 - |J|/2 - 4(m+1)|J|,\ t_0 + |J|/2 + 4(m+1)|J|)$. By taking a sufficiently small C_0 it can be assumed that $|J'| \leqslant \pi/4$.

Now, if $t \in J'$ then inequality (3.5) gives

$$|A(t) - P_n^{(m)} A(t)| \leqslant C_m |J|^{-1/p} \cdot n^{-m-1}. \tag{3.6}$$

For $t \in T \backslash J' \subset T \backslash J$ we have (cf. (1.11))

$$|A(t) - P_n^{(m)} A(t)| = |P_n^{(m)} A(t)|$$
$$\leqslant \sum_{i=1}^{n} |(A, \underline{N}_{n,i}^{(m)})| \cdot N_{n,i}^{(m)}(t). \qquad (3.7)$$

Integrating by part and using (1.9) we obtain the estimate

$$|(A, N_{n,i}^{(m)})| = \left| \int_\alpha^\beta A(t) \cdot N_{n,i}^{(m)}(t)\, dt \right|$$
$$= \left| \int_\alpha^\beta a(t) \int_\alpha^t \frac{(t-\xi)^m}{m!} \cdot N_{n,i}^{(m)}(\xi)\, d\xi\, dt \right|$$
$$\leqslant C_m \cdot \|a\|_\infty \cdot \int_\alpha^\beta n \cdot \frac{(\beta - \xi)^{m+1}}{(m+1)!} \cdot q^{n \cdot d_T(\xi, s_{n,i})}\, d\xi.$$

Because of $q < 1$, for $s'_{n,i} \in (t_0 + |J|/2, t_0 + \pi]$ from this relation it easily follows that

$$|(A, N_{n,i}^{(m)})| \leqslant C_m \cdot n^{-m-1} \cdot |J|^{-1/p} \cdot q_1^{n \cdot d_T(J, s_{n,i})} \qquad (3.8)$$

where again $0 < q_1 < 1$. This inequality also holds in the case $s'_{n,i} \in (t_0 - \pi, t_0 - |J|/2)$ (employ the analogous estimate

$$|(A, N_{n,i}^{(m)})| \leqslant C_m \|a\|_\infty \int_\alpha^\beta n \cdot (\xi - \alpha)^{m+1} \cdot q^{n \cdot d_T(s_{n,i}, \xi)}\, d\xi).$$

For given $t \in T \backslash J'$ it can easily be checked that by the definition of the B-splines (cf. (1.7)) the relation $N_{n,i}^{(m)}(t) \neq 0$ implies

$$|t - s'_{n,i}| < 4\pi \cdot (m+1)/n < \tfrac{1}{2} d_T(t, J)$$

and, thus,

$$d_T(s_{n,i}, J) \geqslant d_T(t, J) - 4\pi \cdot (m+1)/n \geqslant \tfrac{1}{2} d_T(t, J).$$

But by (1.7) we have $N_{n,i}^{(m)}(t) \neq 0$ for $m+1$ values of i at most. Therefore, (3.7) and (3.8) yield

$$|A(t) - P_n^{(m)} A(t)| \leqslant C_m n^{-m-1} |J|^{-1/p} \cdot q_2^{n \cdot d_T(t, J)}, \qquad t \in T \backslash J'. \quad (3.9)$$

Here, $0 < q_2 = q_1^{1/2} < 1$. The estimates (3.6) and (3.9) are sufficient for our purposes.

Let

$$\beta_l = \int_{t_0-\pi}^{t_0+\pi} (A(t) - P_n^{(m)} A(t)) \cdot (t-t_0)^l \, dt, \qquad l = 0,..., [1/p - 1],$$

be the moments of $A(t) - P_n^{(m)} A(t)$. Obviously, $\beta_0 = 0$. For sufficiently large n_0 (depending only on m) we can determine spline functions $\varphi_l(t) \in S_{n_0}^{(m)}(T)$ satisfying

$$\varphi_l(t) = (t-t_0)^l, \qquad t \in (t_0 - \pi/2, t_0 + \pi/2), \qquad \|\varphi_l\|_\infty \leqslant C_m$$

for $l = 0,..., [1/p - 1]$. Due to the properties of $A(t)$, $\varphi_l(t)$, and (3.9) we obtain

$$\begin{aligned} |\beta_1| &\leqslant \int_{t_0-\pi}^{t_0+\pi} |A(t) - P_n^{(m)} A(t)| \cdot |(t-t_0)^l - \varphi_l(t)| \, dt \\ &\quad + \left| \int_T (A(t) - P_n^{(m)} A(t)) \cdot \varphi_l(t) \, dt \right| \\ &\leqslant C_m \cdot n^{-m-1} |J|^{-1/p} q_2^{n\pi/8} \leqslant C_m \cdot n^{-3m-3} |J|^{-1/p}, \quad n \geqslant n_0, \end{aligned} \tag{3.10}$$

since

$$\begin{aligned} &\int_T P_n^{(m)} A(t) \cdot \varphi_l(t) \, dt \\ &\quad = \int_T A(t) \cdot P_n^{(m)} \varphi_l(t) \, dt = \int_A A(t) \cdot \varphi_l(t) \, dt \\ &\quad = \int_J A(t) \cdot (t-t_0)_0^l \, dt = 0, \qquad l = 0,..., [1/p - 1], n \geqslant n_0. \end{aligned}$$

Let $\psi_j(t)$, $j = 0,..., [1/p - 1]$, be the unique set of polynomials of order up to $[1/p - 1]$ satisfying

$$\int_{t_0-|J|/2}^{t_0+|J|/2} \psi_j(t) \cdot (t-t_0)^l \, dt = \delta_{jl}, \quad j, l = 0,..., [1/p - 1].$$

Obvious, $\|\psi_j\|_\infty \leqslant C_m \cdot |J|^{-j-1} \leqslant C_m \cdot n^{m+1}$, and therefore, for the function

$$\begin{aligned} B(t) &= 0, & t &\in T \backslash J, \\ &= \sum_{j=0}^{[1/p-1]} \beta_j \cdot \psi_j(t), & t &\in J, \end{aligned}$$

we get with (3.10)

$$\|B\|_{\mathrm{Re}\,H_p(T)} \leqslant C \cdot \|B\|_\infty \leqslant C_m n^{-2m-2} \cdot |J|^{-1/p} \leqslant C_m \cdot n^{-m-1}. \quad (3.11)$$

Furthermore, by (3.6), (3.9) for $m(t) = A(t) - P_n^{(m)} A(t) - B(t)$ we obtain

$$\begin{gathered} |m(t)| \leqslant C_m \cdot |J|^{-1/p} \cdot n^{-m-1} \cdot q_2^{n \cdot d_T(t, J')}, \\ \int_{t_0-\pi}^{t_0+\pi} m(t) \cdot (t-t_0)^l \, dt = \beta_l - \beta_l = 0, \quad l = 0, \ldots, [1/p - 1]. \end{gathered} \quad (3.12)$$

Equation (3.12) yields that $m(t)$ is a $(p, \infty, [1/p-1])$-molecule centered at t_0 because (cf. (1.5))

$$\begin{aligned} N(m) &\leqslant C_m \cdot |J|^{-1/p} \cdot n^{-m-1} \cdot \|q_2^{n \cdot d_T(t, J')} \cdot d_T(t, t_0)\|_\infty^{1-\alpha/\beta} \\ &\leqslant C_m \cdot |J|^{-1/p} \cdot n^{-m-1} \cdot |J|^{1/p} = C_m n^{-m-1} < \infty. \end{aligned}$$

By Proposition 2 and (3.11) we finally obtain

$$\begin{aligned} \|A(t) - P_n^{(m)} A(t)\|_{\mathrm{Re}\,H_p(T)} &\leqslant C_{m,p} \{N(m) + \|B\|_{\mathrm{Re}\,H_p(T)}\} \\ &\leqslant C_{m,p} n^{-m-1}. \end{aligned}$$

Thus, (3.3′) is established (the independence on p of the constant easily follows by interpolating the endpoint-estimates for $p = (m+1)^{-1}$ and $p = 1$ or by using $(p, \infty, [1/p-1] + m + 2)$-atoms instead of $(p, \infty, [1/p-1] + m + 1])$-atoms), and the proof of Theorem 2 is complete.

Remark 3. The inequality

$$\|f - P_n^{(m)} f\|_p \leqslant C_m \cdot \omega_{m+1}(\pi/n, f)_p, \qquad n = 1, 2, \ldots, \quad (3.13)$$

where $f(t) \in L_p(T)$, $1 \leqslant p < \infty$, was essentially proved by Z. Ciesielski [3]. For this case he also stated some inverse inequalities (cf. [3, Sect. 9]). Let

$$E_n^{(m)}(u)_{\mathrm{Re}\,H_p} = \inf_{g(t) \in S_n^{(m)}(T)} \|u - g\|_{\mathrm{Re}\,H_p(T)}, \qquad n = 1, 2, \ldots,$$

be the best spline approximation of $u(t) \in \mathrm{Re}\,H_p(T)$ with respect to $S_n^{(m)}(T)$. Equation (3.2) is equivalent to the Jackson-type inequality

$$E_n^{(m)}(u)_{\mathrm{Re}\,H_p} \leqslant C_m \cdot \omega_{m+1}(\pi/n, u)_{\mathrm{Re}\,H_p}, \qquad n = 1, 2, \ldots, \quad (3.14)$$

where $u(t) \in \mathrm{Re}\,H_p(T)$ and $(m+1)^{-1} \leqslant p \leqslant 1$. In the case $0 < p < (m+1)^{-1}$ as well as concerning inverse inequalities no results seem to be known at present.

However, for $m=0$ and $\frac{1}{2} < p < 1$ we are able to state the inequalities $(n = 1, 2, \ldots)$

$$C_p \cdot E_n^{(0)}(u)_{\operatorname{Re} H_p} \leqslant \omega_1(1/n, u)_{\operatorname{Re} H_p} \tag{3.15}$$
$$\leqslant C_p' \cdot n^{-1} \left\{ \sum_{k=1}^{n} E_k^{(0)}(u)_{\operatorname{Re} H_p}^p \cdot k^{p-1} \right\}^{1/p}, u(t) \in \operatorname{Re} H_p(T).$$

The first part of (3.15) follows by observing that the restrictions $a_i(t) = P_n^{(1)} u(t) - P_n^{(0)} P_n^{(1)} u(t)|_{t \in (s_{n,i-1}, s_{n,i})}$, $i = 1, \ldots, n$, have the properties of $(p, \infty, 0)$-atoms, $\frac{1}{2} < p \leqslant 1$:

$$\operatorname{supp} a_i(t) = (s_{n,i-1}, s_{n,i}) = \Delta_i, \qquad |\Delta_i| \asymp n^{-1},$$
$$\|a_i\|_\infty = \tfrac{1}{2} \cdot |P_n^{(1)} u(s_{n,i}) - P_n^{(1)} u(s_{n,i-1})| = \tfrac{1}{2} \cdot d_i,$$
$$\int_{\Delta_i} a_i(t)\, dt = 0, \qquad i = 1, \ldots, n.$$

Thus, by Proposition 1(b)

$$\|P_n^{(1)} u - P_n^{(0)} P_n^{(1)} u\|_{\operatorname{Re} H_p(T)} \leqslant C_p \cdot n^{-1/p} \left\{ \sum_{i=1}^{n} d_i^p \right\}^{1/p}.$$

But $|\Delta^1_{\pi \cdot 2^{1-k}} P_{2^k}^{(1)} u(s_{2^k,i})| = d_{i-1}$, $i = 1, \ldots, 2^k$, and $\Delta^1_{\pi \cdot 2^{1-k}} P_{2^k}^{(1)} u(t) \in S_{2^k}^{(1)}(T)$, hence we have

$$\|P_{2^k}^{(1)} u - P_{2^k}^{(0)} P_{2^k}^{(1)} u\|_{\operatorname{Re} H_p(T)}$$
$$\leqslant C_p \left\{ \sum_{i=1}^{2^k} 2^{-k} d_i^p \right\}^{1/p} \leqslant C_p \cdot \|\Delta^1_{\pi \cdot 2^{1-k}} P_{2^k}^{(1)} u\|_p$$
$$\leqslant C_p \cdot \omega_1(2^{-k}, P_{2^k}^{(1)} u)_{\operatorname{Re} H_p}, \quad k = 0, 1, \ldots.$$

Thus, according to Theorem 2 with $m=1$, $\frac{1}{2} < p < 1$, for $n = 2^{+k}, \ldots, 2^{k+1} - 1$ $(k = 0, 1, \ldots,)$ we obtain the estimates

$$E_n^{(0)}(u)_{\operatorname{Re} H_p}$$
$$\leqslant E_{2^k}^{(0)}(u)_{\operatorname{Re} H_p} \leqslant \|u - P_{2^k}^{(0)} P_{2^k}^{(1)} u\|_{\operatorname{Re} H_p(T)}$$
$$\leqslant C_p \{ \|u - P_{2^k}^{(1)} u\|_{\operatorname{Re} H_p(T)} + \|P_{2^k}^{(1)} u - P_{2^k}^{(0)} P_{2^k}^{(1)} u\|_{\operatorname{Re} H_p(T)} \}$$
$$\leqslant C_p \{ \omega_2(2^{-k}, u)_{\operatorname{Re} H_p} + \omega_1(2^{-k}, u - P_{2^k}^{(1)} u)_{\operatorname{Re} H_p} + \omega_1(2^{-k}, u)_{\operatorname{Re} H_p} \}$$
$$\leqslant C_p \cdot \omega_1(1/n, u)_{\operatorname{Re} H_p}.$$

This proves the Jackson-type inequality in (3.15).

In order to verify the inverse inequality we consider an arbitrary step function $g(t) \in S_{2^k}^{(0)}(T)$ and $0 < h \leqslant \pi \cdot 2^{1-k}$. Let

$$g_i(t) = g(t)|_{(s_{2^k,i}, s_{2^k,i+1})}, \qquad i = 1,..., 2^k.$$

After suitable normalization the functions $\Delta_h^1 g_i(t)$ can be treated as $(p, 1, 0)$-atoms $(\frac{1}{2} < p < 1)$ because we have

$$\operatorname{supp} \Delta_h^1 g_i(t) \subset \Delta_i' = (s_{2^k,i-1}, s_{2^k,i+1}), \quad |\Delta_i'| \asymp 2^{-k},$$

$$\|\Delta_h^1 g_i\|_1 = 2h \cdot \|g_i\|_\infty, \int_{\Delta_i'} \Delta_h^1 g_i(t)\, dt = 0, \quad i = 1,..., n.$$

Therefore (cf. (1.1) and Proposition 1(b)),

$$\begin{aligned} \|\Delta_h^1 g\|_{\operatorname{Re} H_p(T)} &\leqslant C_p \cdot n \cdot h \left\{ \sum_{i=1}^n 1/n \cdot \|g_i\|_\infty^p \right\}^{1/p} \leqslant C_p \cdot n \cdot h \cdot \|g\|_p \\ &\leqslant C_p \cdot n \cdot h \cdot \|g\|_{\operatorname{Re} H_p(T)}, \qquad 0 < h < 2\pi/n, \quad \tfrac{1}{2} < p < 1, \quad (3.16) \end{aligned}$$

where $n = 2^k$, $k = 0, 1,...$. This inequality plays the role of a Bernstein-type estimate.

Relation (3.16) immediately yields

$$\begin{aligned} &\|\Delta_h^1 u\|_{\operatorname{Re} H_p(T)}^p \\ &\quad\leqslant \|\Delta_h^1(u - \bar{g}_{2^k})\|_{\operatorname{Re} H_p(T)}^p + \sum_{j=1}^k \|\Delta_h^1(\bar{g}_{2^j} - \bar{g}_{2^{j-1}})\|_{\operatorname{Re} H_p(T)}^p \\ &\quad\leqslant C_p \left\{ \|u - \bar{g}_{2^k}\|_{\operatorname{Re} H_p(T)}^p + \sum_{j=1}^k (2^j \cdot h)^p \|\bar{g}_{2^j} - \bar{g}_{2^{j-1}}\|_{\operatorname{Re} H_p(T)}^p \right\} \\ &\quad\leqslant C_p \cdot h^p \cdot \sum_{j=0}^k 2^{jp} \cdot E_{2^j}^{(0)}(u)_{\operatorname{Re} H_p}^p, \quad 0 < h < \pi \cdot 2^{1-k}, \ k = 0, 1,..., \end{aligned}$$

where $\bar{g}_n(t) \in S_n^{(0)}(T)$, $n = 1, 2,...$, are the best approximating step functions, and easy computations give the second part of (3.15).

Probably, the real methods presented here can be used to settle the general case, too. For instance, by analogous considerations the inverse inequality

$$\omega_{m+1}(1/n, u)_{\operatorname{Re} H_p} \leqslant C_{m,p} \cdot n^{-m-1} \left\{ \sum_{k=1}^k k^{(m+1)p-1} \cdot E_k^{(m)}(u)_{\operatorname{Re} H_p}^p \right\}^{1/p},$$

$n = 1, 2,..., u(t) \in \operatorname{Re} H_p(T)$, can be established for arbitrary $(m+2)^{-1} < p < 1$ and $m = 0, 1,...$.

Finally, it should be mentioned that the inverse inequalities stated here differ from the corresponding estimates for the L_p-spaces, $0 < p < 1$, considered in [11].

Remark 4. Here the nonperiodic case will briefly be considered. The definitions and basic properties of the Hardy spaces defined on the real line $\mathbb{R}$ are quite similar to the periodic case (cf. [8] for the classical Hardy space of analytic functions on the upper half-plane, and [5, 27] concerning the atomic decompositions).

THEOREM 1′. *Let $f(x) \in \operatorname{Re} H_p(\mathbb{R})$, $0 < p \leqslant 1$, and $k = 1, 2, \ldots$. Then we have*

$$\omega_k(\delta, f)_{\operatorname{Re} H_p(\mathbb{R})} = \sup_{0 < h \leqslant \delta} \|\Delta_h^k f(x)\|_{\operatorname{Re} H_p(\mathbb{R})}$$

$$\overset{k,p}{\asymp} \inf_{D^k g(x) \in \operatorname{Re} H_p(\mathbb{R})} \{\|f - g\|_{\operatorname{Re} H_p(\mathbb{R})}$$

$$+ \|D^k g\|_{\operatorname{Re} H_p(\mathbb{R})} \cdot \delta^k\}, \tag{3.17}$$

where $0 < \delta < \infty$ and the derivative $D^k g(t)$ has to be understood in the sense of $S'(\mathbb{R})$.

The proof is analogous to that of Theorem 1 and will be omitted. In the following we shall concentrate on an application of (3.17) to approximation estimates for Bochner–Riesz summability. Let $\delta > 0$, $R > 0$, and

$$S_R^\delta f(x) = \int_{\mathbb{R}} (1 - y^2/R^2)_+^\delta \cdot \hat{f}(y) \cdot e^{2\pi i x \cdot y}\, dy$$

$$= \pi^{-\delta} \cdot \Gamma(\delta + 1) \cdot \int_{\mathbb{R}} f(x - u/R)$$

$$\cdot |u|^{-1/2-\delta} \cdot J_{1/2+\delta}(|u|)\, du \tag{3.18}$$

be the corresponding Bochner–Riesz means of the Fourier integral of $f(x)$, where $J_\alpha(s)$ and $\hat{f}(y)$ denote the Bessel function of order α and the Fourier transform of $f(x)$, resp. Obviously, the first expression in (3.18) makes sense for arbitrary $f(x) \in \operatorname{Re} H_p(\mathbb{R})$ while the second one holds for $f(x) \in L_1(\mathbb{R})$ at least (cf. [9, 19]). From the results of [16, 20] it follows that in the case $\delta > 1/p - 1$ the relations

$$\sup_{R > 0} \|S_R^\delta f\|_{\operatorname{Re} H_p(\mathbb{R})} \leqslant C_{p,\delta} \|f\|_{\operatorname{Re} H_p(\mathbb{R})} \tag{3.19}$$

and

$$\lim_{R \to \infty} \|f - S_R^\delta f\|_{\operatorname{Re} H_p(\mathbb{R})} = 0$$

hold for arbitrary $f(x) \in \operatorname{Re} H_p(\mathbb{R})$, $0 < p \leqslant 1$.

We want to state a somewhat stronger result.

THEOREM 3. *Let* $f(x) \in \operatorname{Re} H_p(\mathbb{R})$, $0 < p \leqslant 1$, *and* $\delta > 1/p - 1$. *Then for* $R > 0$

$$\|f - S_R^\delta f\|_{\operatorname{Re} H_p(\mathbb{R})} \leqslant C_{p,\delta}\, \omega_1(1/R, f)_{\operatorname{Re} H_p(\mathbb{R})}. \tag{3.20}$$

Proof. It suffices to prove the estimate

$$\|A - S_R^\delta A\|_{L_p(\mathbb{R})} \leqslant C_{p,\delta} \cdot R^{-1}, \qquad R > 0, \quad \delta > 1/p - 1, \tag{3.21}$$

for arbitrary $(p, \infty, [1/p - 1] + 1)$-atoms $a(x)$, where $A(x) = \int_{-\infty}^x a(y)\, dy$. The rest will follow by standard arguments from the H_p-theory (cf. below). By definition

$$\begin{gathered} \operatorname{supp} a(x) \subset J = (t_0 - |J|/2, t_0 + |J|/2), \qquad t_0 \in \mathbb{R}, \quad |J| < \infty, \\ \|a\|_{L_\infty(\mathbb{R})} \leqslant |J|^{-1/p}, \\ \int_J a(x) \cdot x^l\, dx = 0, \qquad l = 0, \ldots, [1/p], \end{gathered} \tag{3.22}$$

and, therefore, $A(x)$ satisfies the properties $\operatorname{supp} A(x) \subset J$, $\|A\|_{L_\infty(\mathbb{R})} \leqslant |J|^{-1/p+1}$, and $\int_J A(x) \cdot x^l\, dx = 0$, $l = 0, \ldots, [1/p - 1]$. Thus, by Coifman's theorem [5] and (3.19) we obtain the estimate

$$\|A - S_R^\delta A\|_{\operatorname{Re} H_p(\mathbb{R})} \leqslant C_{p,\delta}\, \|A\|_{\operatorname{Re} H_p(\mathbb{R})} \leqslant C_{p,\delta} |J|,$$

which yields (3.21) for $|J| \leqslant 1/R$.

On the other hand, from (3.18) we obtain

$$\begin{aligned} &|A(x) - S_R^\delta A(x)| \\ &\quad \leqslant C_\delta \cdot \left| \int_{\mathbb{R}} (A(x) - A(x - u/R)) \cdot |u|^{-1/2-\delta} \cdot J_{1/2+\delta}(|u|)\, du \right| \\ &\quad \leqslant C_\delta \cdot \|a\|_{L_\infty(\mathbb{R})} \int_J \left| \int_{R|x-y|} u^{-1/2-\delta} \cdot J_{1/2+\delta}(u)\, du \right| dy. \end{aligned}$$

According to the asymptotic behaviour of the Bessel functions

$$\begin{aligned} J_\alpha(u) &= 0(u^\alpha), \qquad u \to 0+, \\ &= C_\alpha \cdot u^{-1/2} \cdot \cos(u - (\alpha + 1/2) \cdot \pi/2) + 0(u^{-3/2}), \quad u \to +\infty \end{aligned}$$

(cf. [19]), we get for $\delta > 0$

$$I_\delta(s) = \left| \int_s^\infty u^{-1/2-\delta} \cdot J_{1/2+\delta}(u)\, du \right| \leqslant C_\delta \cdot \min(1, s^{-1-\delta}),\ s > 0.$$

This obviously yields the pointwise estimate

$$|A(x) - S_R^\delta A(x)| \leqslant C_\delta |J|^{-1/p} \begin{cases} 1/R, & |x-t_0| \leqslant 2|J|, \\ |J| \cdot (R|x-t_0|)^{-1-\delta}, & |x-t_0| > 2|J|, \end{cases} \tag{3.24}$$

for the case $|J| > 1/R$. From (3.24) it immediately follows that

$$\begin{aligned} \|A - S_R^\delta A\|_{L_p(\mathbb{R})} &\leqslant C_{p,\delta} |J|^{-1/p} \left\{ \int_{|x-t_0| \leqslant 2|J|} R^{-p}\, dx \right. \\ &\quad \left. + |J|^p R^{-(1+\delta)p} \int_{2 \cdot |J|}^\infty \frac{ds}{s^{(1+\delta)p}} \right\}^{1/p} \\ &\leqslant C_{p,\delta} \cdot R^{-1} \cdot \{1 + |J|^{p-1} \cdot R^{-\delta p} \cdot |J|^{1-p-\delta p}\}^{1/p} \\ &\leqslant C_{p,\delta} \cdot R^{-1}, \qquad \delta > 1/p - 1, \quad |J| > 1/R, \end{aligned}$$

and (3.21) is completely proved.

Considering any $g(x) \in \mathscr{S}'(\mathbb{R})$ with $D^1 g(x) \in \operatorname{Re} H_p(\mathbb{R})$, let

$$g(x) = \sum_{j=1}^\infty \lambda_j \cdot A_j(x), \qquad D^1 g(x) = \sum_{j=1}^\infty \lambda_j \cdot a_j(x),$$

be the corresponding atomic decompositions (here $a_j(x)$ are $(p, \infty, [1/p])$-atoms (cf. (3.22)), $A_j(x) = \int_{-\infty}^x a_j(y)\, dy$, $j = 1, 2, \ldots$, and

$$\left\{ \sum_{j=1}^\infty |\lambda_j|^p \right\}^{1/p} \overset{p}{\asymp} \|D^1 g\|_{\operatorname{Re} H_p(\mathbb{R})}).$$

Then according to (3.21) we have

$$\begin{aligned} \|g - S_R^\delta g\|_{L_p(\mathbb{R})}^p &\leqslant \sum_{j=1}^\infty |\lambda_j|^p \|A_j - S_R^\delta A_j\|_{L_p(\mathbb{R})}^p \\ &\leqslant C_{p,\delta} \cdot R^{-p} \cdot \sum_{j=1}^\infty |\lambda_j|^p \leqslant C_{p,\delta} \{R^{-1} \cdot \|D^1 g\|_{\operatorname{Re} H_p(\mathbb{R})}\}^p. \end{aligned}$$

Furthermore, let $\tilde{A}_j(x)$ denote the Hilbert transform of $A_j(x)$. Then by the classical definition of $\operatorname{Re} H_p(\mathbb{R})$ we have

$$\|A_j - S_{\mathbb{R}}^\delta A_j\|_{\operatorname{Re} H_p(\mathbb{R})} \leqslant C_p \{ \|A_j - S_R^\delta A_j\|_{L_p(\mathbb{R})} + \|\tilde{A}_j - \widetilde{S_R^\delta A_j}\|_{L_p(\mathbb{R})} \}.$$

But $\widetilde{S_R^\delta A_j}(x) = S_R^\delta \tilde{A}_j(x)$, and $D^1\tilde{A}_j(x) = \tilde{a}_j(x) \in \operatorname{Re} H_p(\mathbb{R})$(as the Hilbert transform of an atom). Thus, by (3.25) we obtain

$$\|A_j - S_R^\delta A_j\|_{\operatorname{Re} H_p(\mathbb{R})} \leqslant C_{p,\delta} R^{-1} \{\|a_j\|_{\operatorname{Re} H_p(\mathbb{R})} + \|\tilde{a}_j\|_{\operatorname{Re} H_p(\mathbb{R})}\} \leqslant C_{p,\delta} R^{-1},$$

from which it follows by repeating the above considerations that

$$\| g - S_R^\delta g\|_{\operatorname{Re} H_p(\mathbb{R})} \leqslant C_{p,\delta} \cdot R^{-1} \cdot \|D^1 g\|_{\operatorname{Re} H_p(\mathbb{R})}, \qquad R > 0, \qquad (3.26)$$

Together with Theorem 1′ ($k = 1$), (3.19) and (3.26) imply the desired inequality (3.20). The proof of Theorem 3 is complete.

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